

POSITIVE DEHN TWIST EXPRESSIONS FOR SOME NEW INVOLUTIONS IN THE MAPPING CLASS GROUP II

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ABSTRACT. This article is continuation from [1]. The positive Dehn twist expressions for the generalization of the involutions described in [1] are presented. The homeomorphism types of the Lefschetz fibrations they define are determined for several examples.

INTRODUCTION

In [1] the author presented the positive Dehn twist expression for a new set of involutions that are obtained by combining two well known involutions in the mapping class group M_g of a 2-dimensional, closed, compact, oriented surface Σ_g of genus $g > 0$, one of which is the hyperelliptic involution, Figure 1. One can extend these new involutions by gluing them together. It is the purpose of this article to find the positive Dehn twist expressions for these extended involutions and compute the signatures of the symplectic Lefschetz fibrations that they describe.

1. REVIEW OF THE SIMPLE CASE

Let i represent the hyperelliptic (horizontal) involution and s represent the vertical involution as shown in Figure 1.

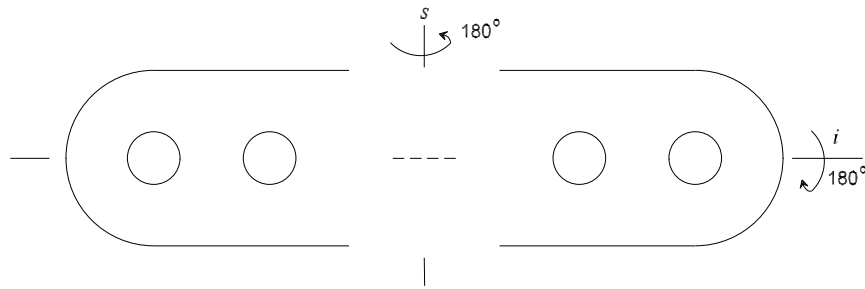


FIGURE 1. The vertical and horizontal involutions

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If i is the horizontal involution on a surface Σ_h and s is the vertical involution on a surface Σ_k , k -even, then let θ be the horizontal involution on the surface Σ_g , where $g = h + k$, obtained as in Figure 2.

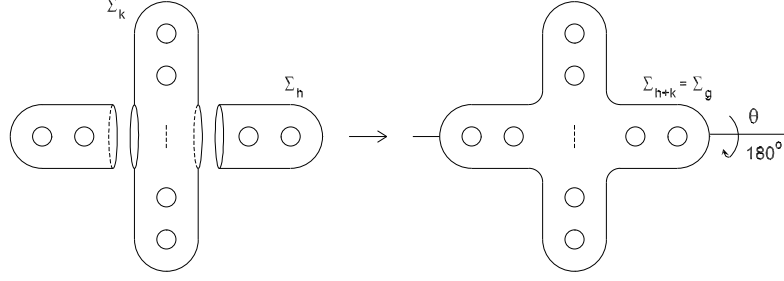


FIGURE 2. The involution θ on the surface Σ_{h+k}

Figure 3 shows the cycles that are used in expressing θ as a product of positive Dehn twists which is stated in the next theorem.

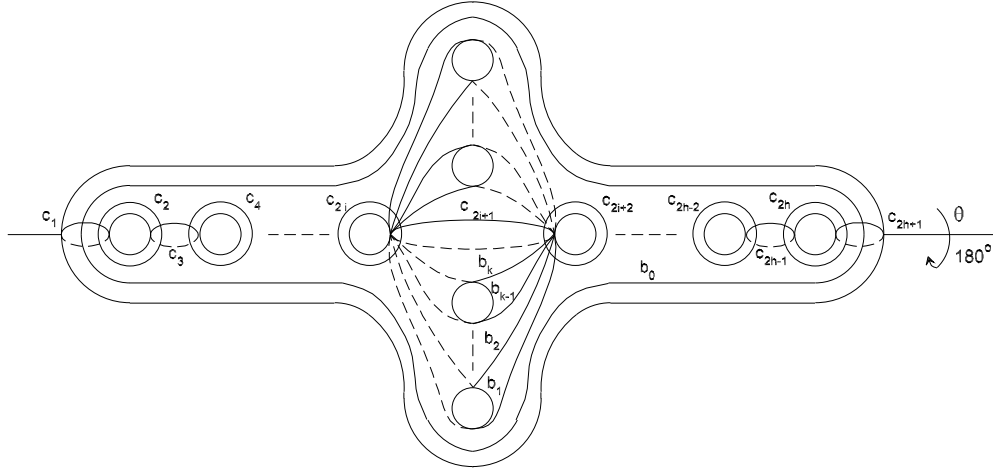


FIGURE 3. The cycles used in the expression of θ

Theorem 1.0.1. *The positive Dehn twist expression for the involution θ shown in Figure 3 is given by*

$$\theta = c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 b_0 c_{2h+1} c_{2h} \cdots c_{2i+2} c_1 c_2 \cdots c_{2i} b_1 b_2 \cdots b_{k-1} b_k c_{2i+1}.$$

See [1] for the proof.

2. MAIN THEOREM

2.1. The Involution θ on bounded surface. Consider the bounded surface $\Sigma_{h+k,2}$ in Figure 4, which is obtained from the surface in Figure 3 by removing a disk from each end. Figure 5 is obtained from Figure 4 by gluing a torus with two boundary components on each end. The cycles shown in Figure 5 realize the involution θ on the bounded surface in Figure 4 as stated in proposition 2.1.2.

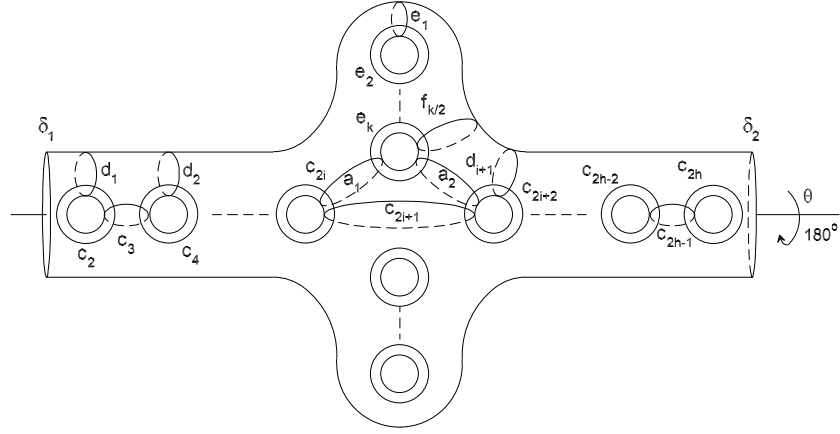


FIGURE 4. The involution θ on bounded surface and a pants decomposition

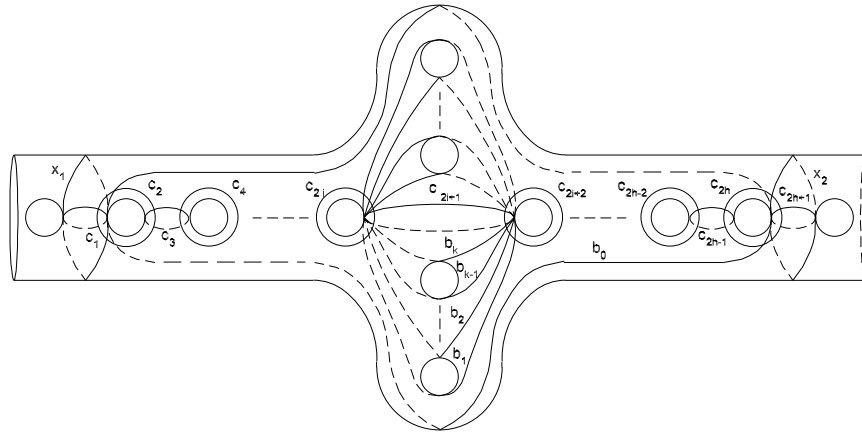


FIGURE 5. The cycles realizing θ on the bounded surface in Figure 4

The boundary components of the chosen pants decomposition shown in Figure 4 will constitute the set of cycles that will be mapped in order to prove proposition 2.1.2.

Since the mapping of many of those boundary components will create in the process cycles that will contain a piece of arc similar to the two that are shown in the first column of Figure 6, we will show the mappings of these segments separately once and use their images in the last column of the same figure to avoid repetition, whenever necessary in the proof of the proposition.

Each row in the following lemma shows the mapping of one of the two types of segments that will occur several times in the proof of proposition 2.1.2 as mentioned above.

Lemma 2.1.1. *The action of the Dehn twists x_1c_1 and x_2c_{2h+1} on the arcs shown in the first column of Figure 6 are as shown in the last column of the same figure.*

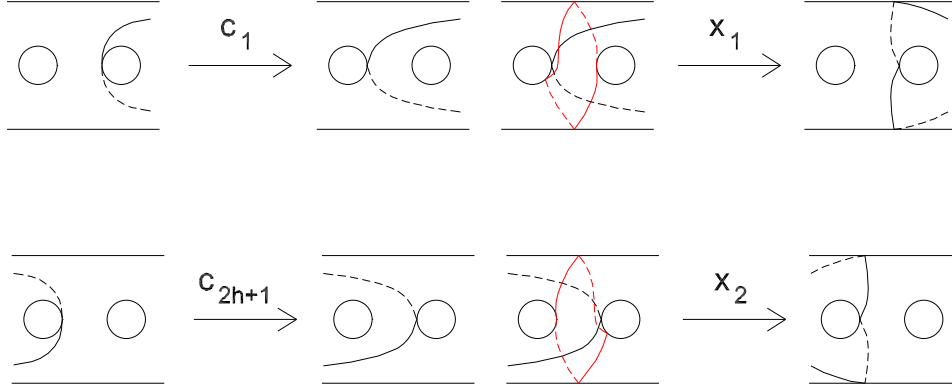


FIGURE 6. The effect of x_1c_1 and x_2c_{2h+1} on the given arcs

Proposition 2.1.2. *The positive Dehn twist expression for the involution θ defined on the bounded surface $\Sigma_{h+k,2}$ shown in Figure 4 is given by*

$$\theta = c_{2i+2} \cdots c_{2h} c_{2i} \cdots c_2 x_2 c_{2h+1} x_1 c_1 b_0 c_{2h} \cdots c_{2i+2} c_2 \cdots c_{2i} b_1 b_2 \cdots b_{k-1} b_k c_{2i+1},$$

where the cycles in the expression are as shown in Figure 5.

Proof: Figure 4 shows a pants decomposition for the bounded surface on which θ is defined. We will show that the given Dehn twist expression in the proposition maps the boundary components of each pair of pants to their images under θ . This will guarantee the mapping of the interior points of each pair of pants accordingly, due to the fact that each twist in the expression is a homeomorphism of the surface onto itself.

The same idea was used in proving theorem 1.0.1 in [1] for the closed surface Σ_{h+k} and the mapping of each boundary cycle was shown there in detail, up to symmetry. Even though the surface subject to this proposition is not closed, there are several figures that are identical for both cases. Therefore, for a given boundary component, instead of repeating verbatim copy of the figures in its mapping from [1], we will skip a few from the beginning and continue from where the different cycles begin to appear. The reader is referred to that article for the details of the mappings that are skipped here.

The boundary components of the chosen pants decomposition in Figure 4 can be summarized as $c_i, i = 1, \dots, 2h + 1, d_i, i = 2, \dots, h - 1, e_i, i = 1, \dots, 2k + 1, f_i, i = 2, \dots, k - 1, a_1, a_2, \delta_1$ and δ_2 along with some additional cycles.

We will begin with the mapping of c_j for j — odd and $2i + 3 \leq j < 2h$, Figure 7. The proof for j — even, including $j = 2$ and $j = 2h$, is similar and was shown in [1]. The mappings of c_{2i+1} and c_{2i+2} will be shown separately.

The long expressions in Figure 7 are due the fact that all the twists they contain miss the cycle that appears in the previous step. The figure shows all the steps there are.

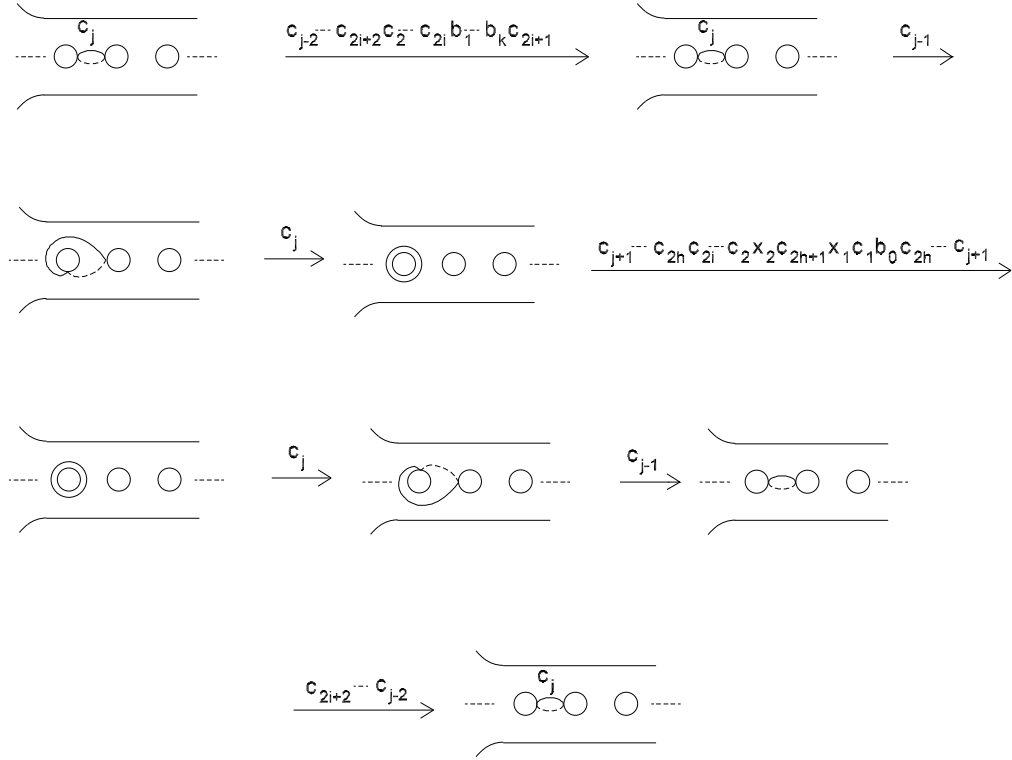


FIGURE 7. The mapping of c_j

We see the mapping of d_j in Figure 8, which is the same for $j = i+1, \dots, h$. The twist about b_0 leaves the curve it is applied to unchanged because their intersection number is 0 as seen in the end of the second line. The result of application of the twists $x_2 c_{2h+1}$ is obtained according to Lemma 2.1.1, therefore only the right end portion of the cycle to which they are applied is modified in the third line. The cycle in the end of the third line is isotopic to the previous one because it is obtained simply by retracting the portion that falls under the surface. The mapping of d_j for $j = 1, \dots, i$ is similar due to symmetry and is omitted.

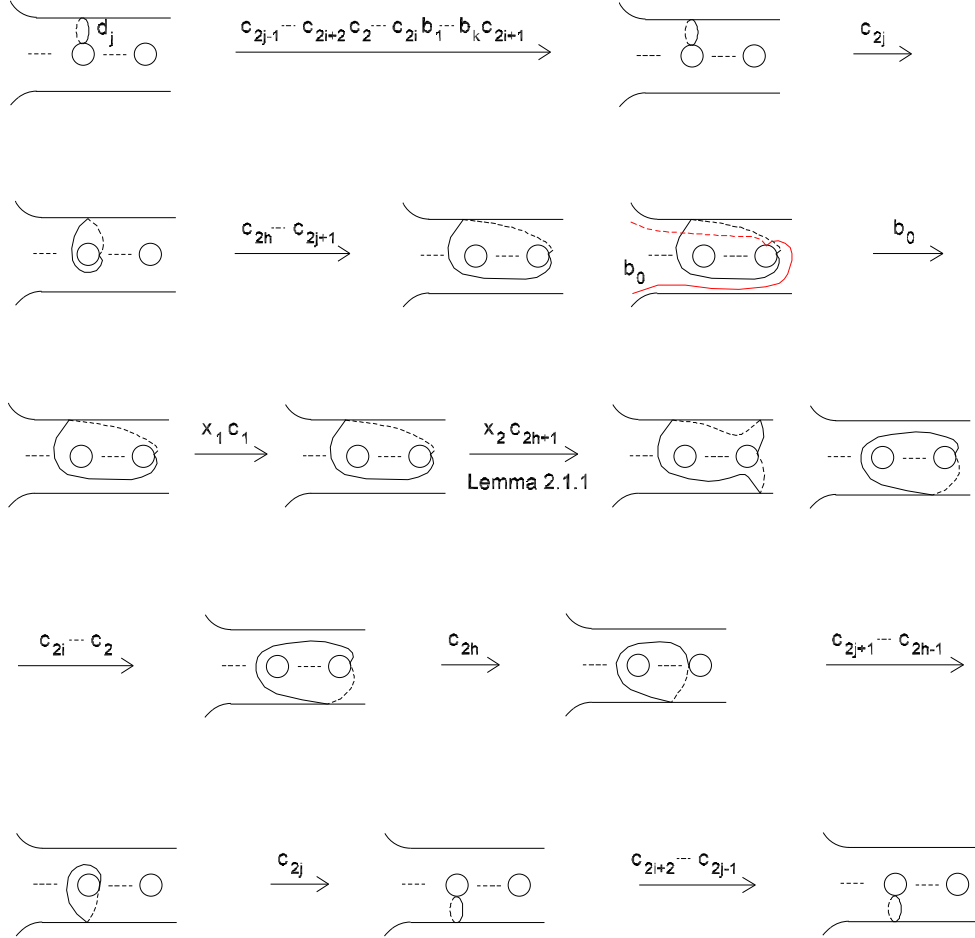
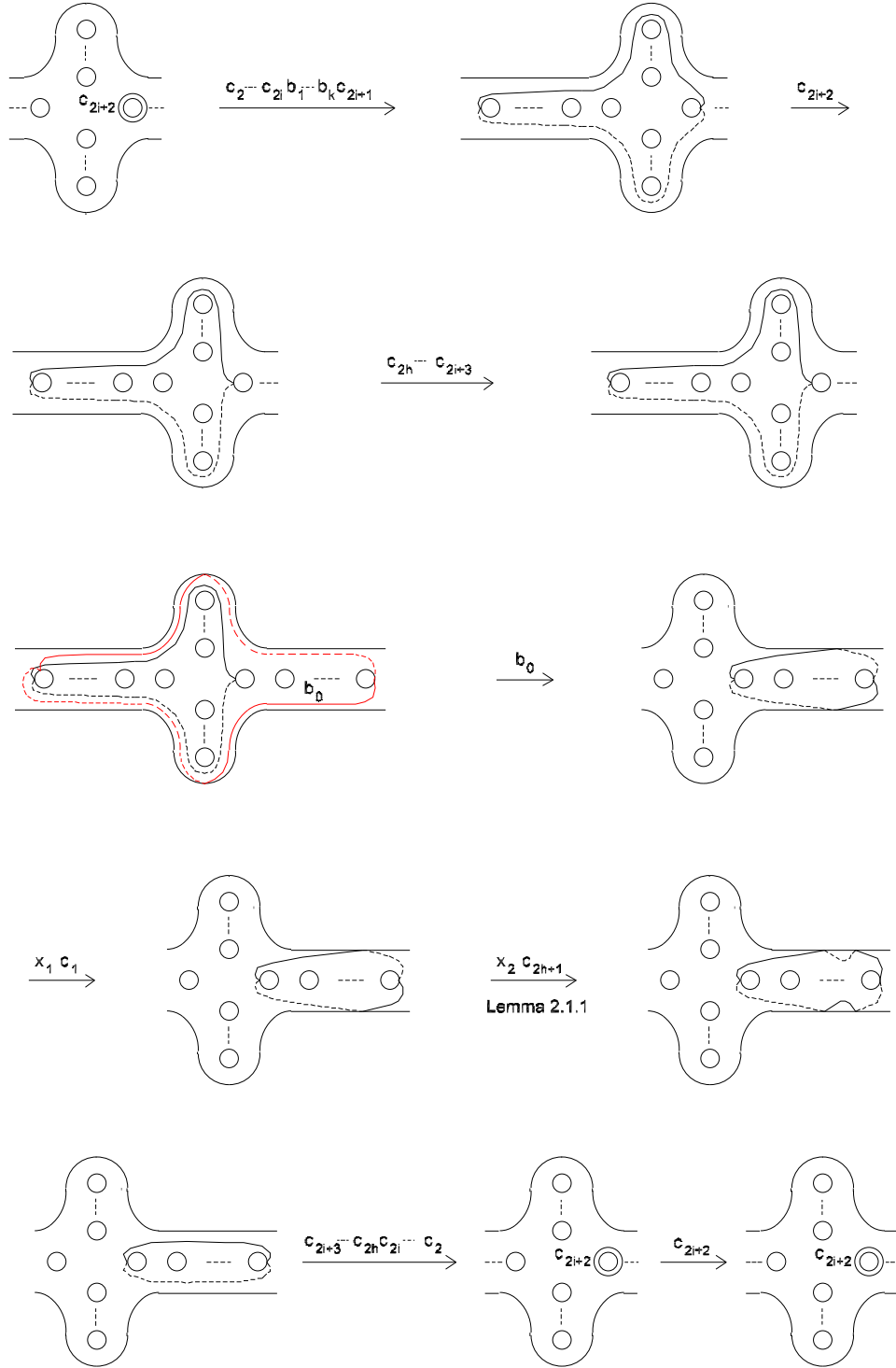


FIGURE 8. The mapping of d_j

Figure 9 shows the mapping of c_{2i+2} . The details of the applications of the twists $c_2 \cdots c_{2i} b_1 \cdots b_k c_{2i+1}$ in the first line are skipped and can be found in [1]. Note the use of Lemma 2.1.1 in the second line from the bottom. The first cycle of the last line is isotopic to the one that appears just before.

FIGURE 9. The mapping of c_{2i+2}

The only curves that are effective in the mappings of e_j are b_j and $b_{j-1}, j = 1, \dots, k$. Figures 10 and 11 show the mapping of e_j for j — odd. The twists in the long expressions all miss the curves that come before them.

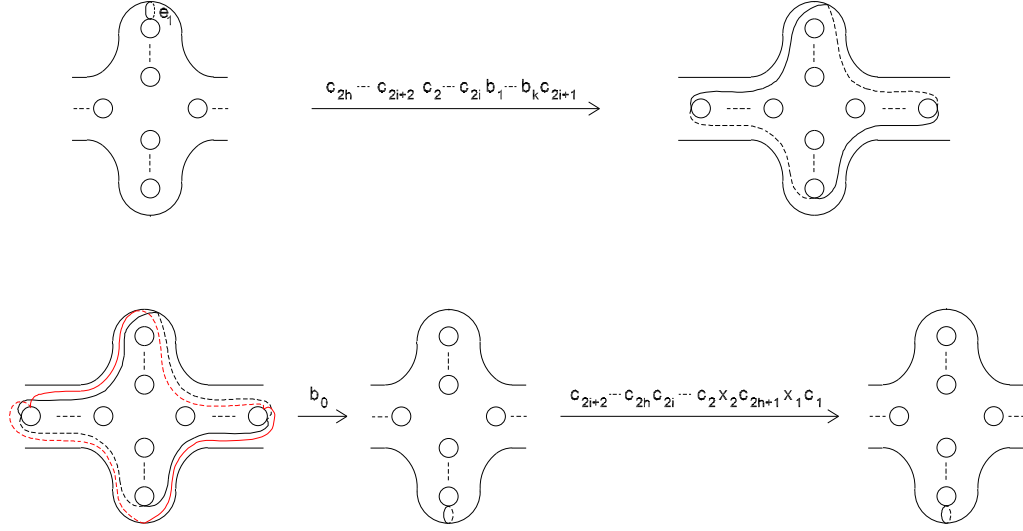


FIGURE 10. The mapping of e_1

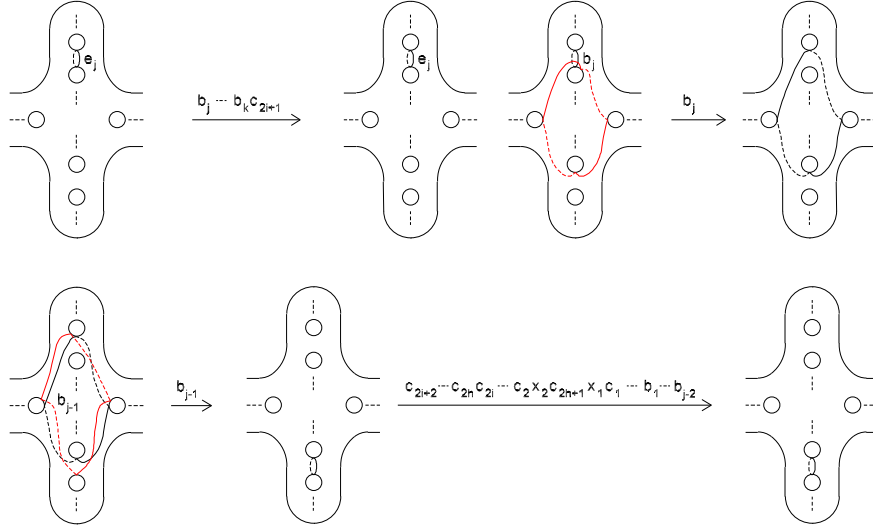


FIGURE 11. The mapping of e_j

The mapping of e_k is a typical example for the mapping of e_j, j — even, which is shown in Figure 12.

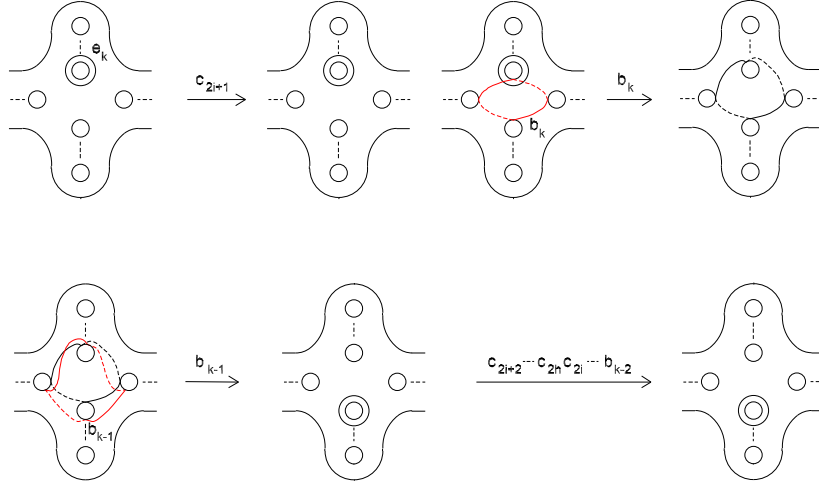


FIGURE 12. The mapping of e_k

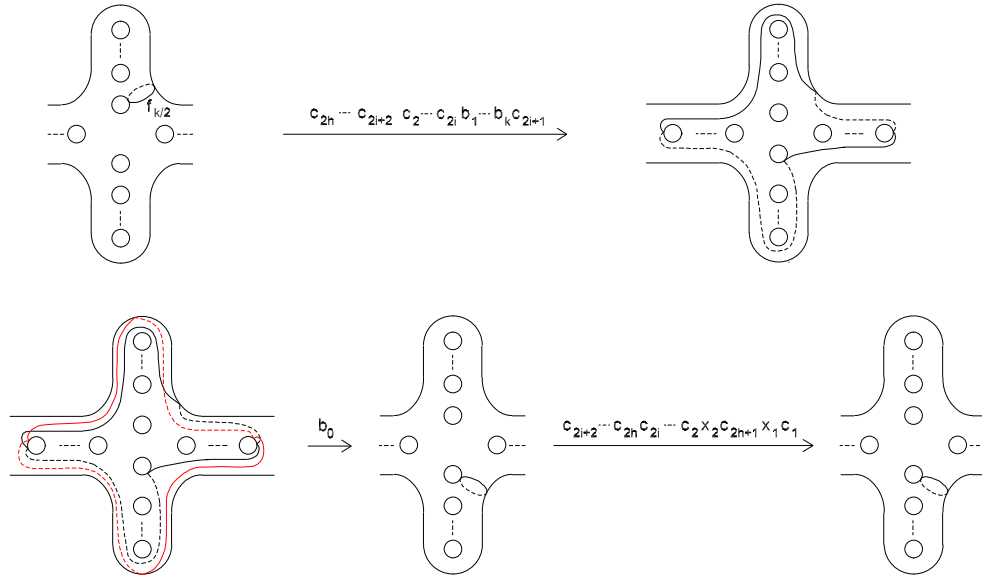


FIGURE 13. The mapping of f_j

The mapping of $f_{k/2}$ is shown in Figure 13. The details of the applications of the twists $c_{2h} \cdots c_{2i+2} c_2 \cdots c_{2i} b_1 \cdots b_k c_{2i+1}$ in the first line are skipped and can be found in [1]. The mappings of f_j for $j = 2, \dots, k/2 - 1$ are similar to that of $f_{k/2}$. Note that f_1 is the same as e_1 .

Figure 14 shows the mapping a_2 and the mapping of a_1 is symmetrical to it. In this figure the details of the applications of the twists $c_{2h} \cdots c_{2i+2} c_2 \cdots c_{2i} b_1 \cdots b_k c_{2i+1}$ are skipped also in the first line. Lemma 2.1.1 is used in the second line and the resulting curve from that is isotopic to the curve in the beginning of the third line.

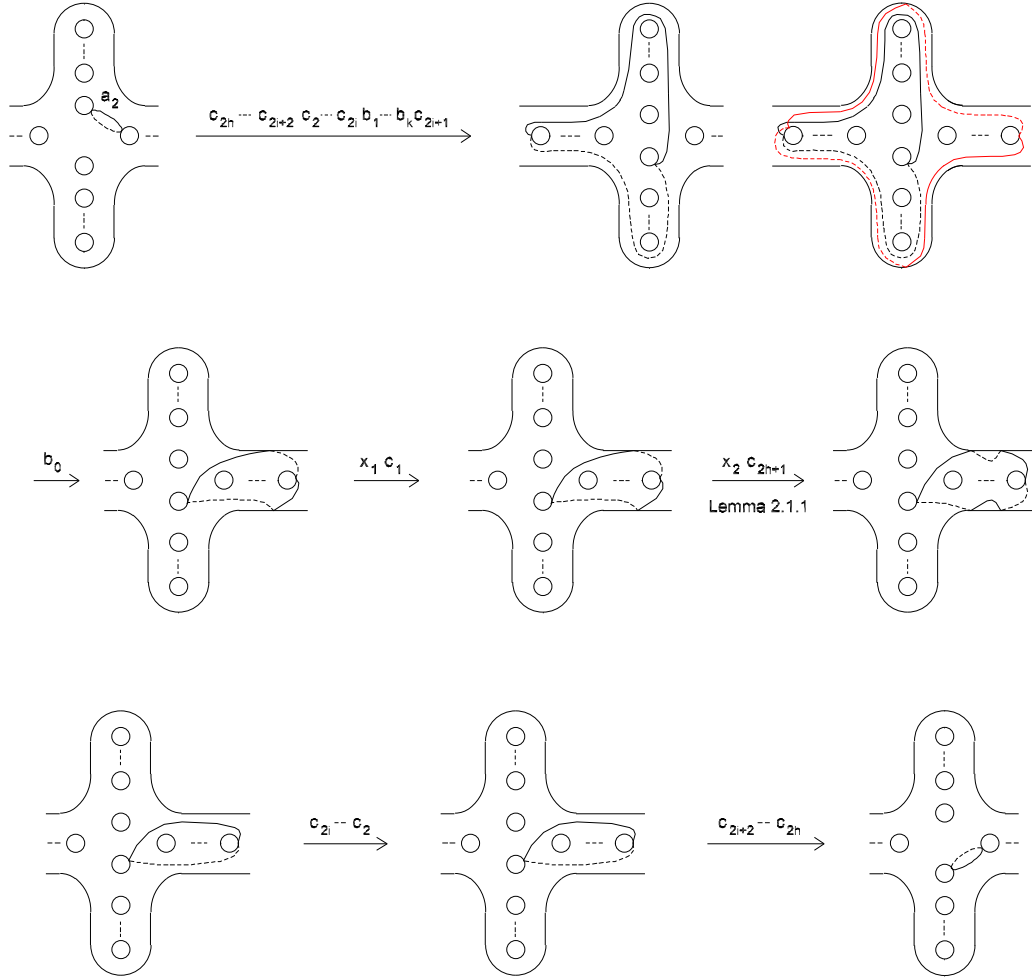
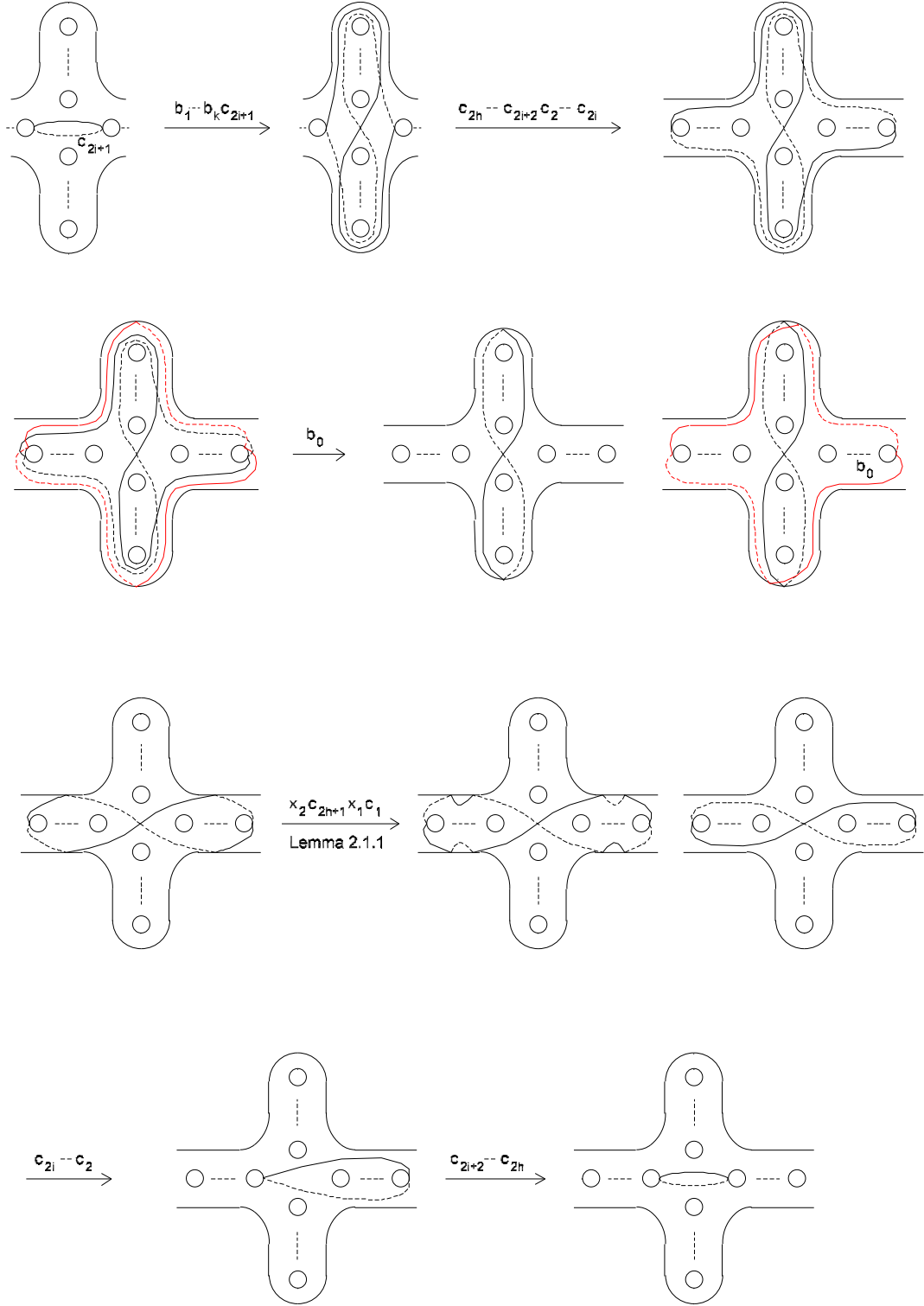


FIGURE 14. The mapping of a_2

In Figure 15 we see the mapping of c_{2i+1} . In this figure, too, the details of the applications of $b_1 \cdots b_k c_{2i+1}$ and $c_{2h} \cdots c_{2i+2} c_2 \cdots c_{2i}$ are skipped in the first line. Lemma 2.1.1 is used twice in the third line and the last figure in that line is isotopic to the one that is resulting from the application of the lemma. Note that b_0 has intersection number 2 with the curve it is applied to; therefore, the result of the twist about b_0 is found by taking their product twice.


 FIGURE 15. The mapping of c_{2i+1}

Finally, we will show the mapping of δ_1 , which is essentially the same as that of δ_2 due to symmetry.

The only cycles that take part in the mapping of δ_1 are c_1 and x_1 , as shown in Figure 16. All the cycles that come before c_1 miss δ_1 as well as the ones that come after x_1 and $x_1 c_1$ fixes δ_1 point-wise. This is shown in Figure 17.

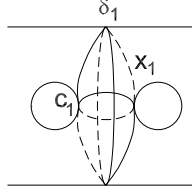


FIGURE 16. δ_1 and the cycles that are effective in its mapping

The intersection number of c_1 and δ_1 is 2; therefore $c_1(\delta_1) = c_1^2 \delta_1$, namely the product of c_1 and δ_1 twice. $c_1(\delta_1)$ is the second cycle in the second row of Figure 17. The intersection number of $c_1(\delta_1)$ and x_1 is also 2; therefore $x_1(t_1(\delta_1)) = x_1^2 t_1(\delta_1)$, which is δ_1 as seen in the last row of the same figure. Therefore $x_1 c_1(\delta_1) = \delta_1$, as claimed.

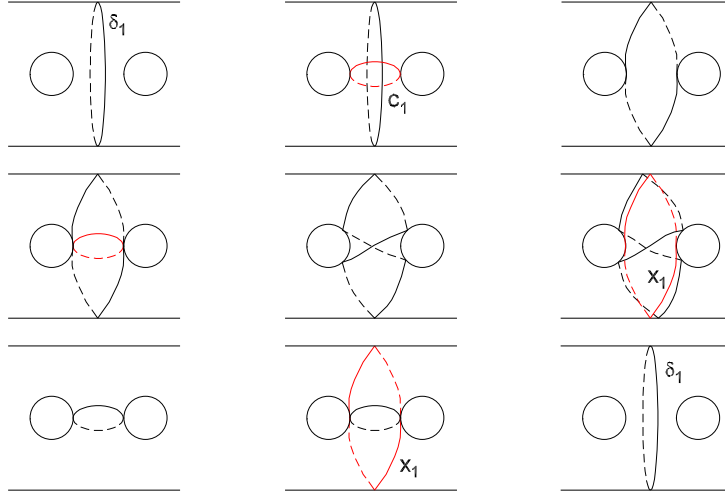


FIGURE 17. The mapping of δ_1

This concludes the proof of Proposition 2.1.2. Now we can prove the main theorem, which is the generalization of Proposition 2.1.2 to a surface that is obtained by gluing n copies of bounded surfaces as in Figure 4 together along four-holed spheres in a sequence. Each copy in that sequence will then have two boundary components except for the first and the last copies, which will have only one boundary component each as shown in Figure 18.

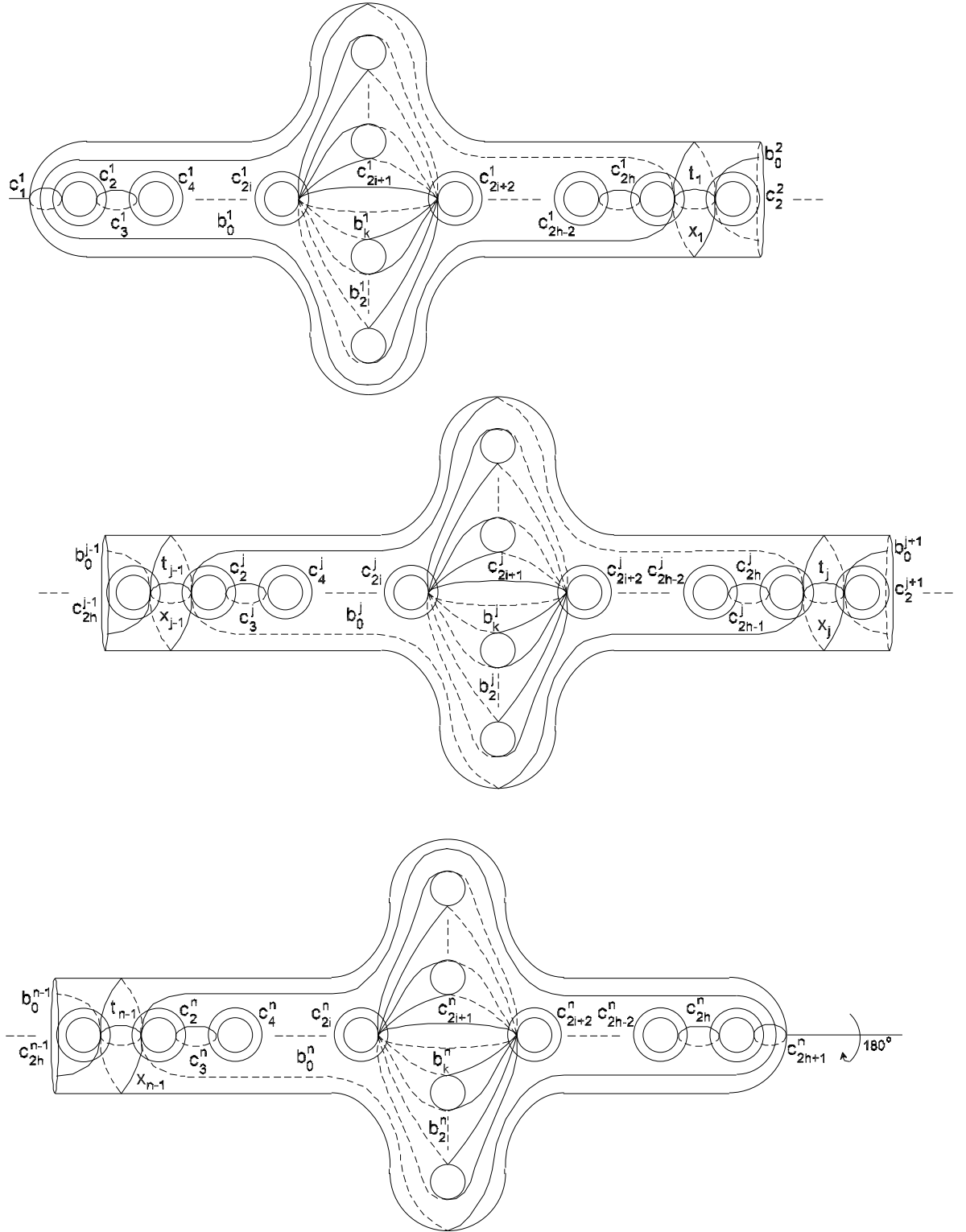


FIGURE 18. General case

In order to simplify the Dehn twist expression for the general case it will be necessary to group the twists in each copy and give suggestive names to them. We will also pay attention to the direction in which the horizontal twists are progressing.

The label of each twist group will carry the following information: Which copy the twist group is in (upper index), whether the group is on the right-hand side or on the left-hand side or in the middle section of the respective copy (name of the group), the direction in which the twists are multiplied when the group consists of horizontal twists (lower index). The twists along the cycle b_0 will not be included in any group. The following is the list of the identifications, except for the first and the last copies:

$$\begin{aligned} l_i^j &= c_{2i}^j \cdots c_2^j, \\ l_o^j &= c_2^j \cdots c_{2i}^j, \\ r_i^j &= c_{2i+2}^j \cdots c_{2h}^j, \\ r_o^j &= c_{2h}^j \cdots c_{2i+2}^j, \\ m^j &= b_1^j \cdots b_k^j c_{2i+1}^j. \end{aligned}$$

The first two lines would be different in the first copy and the third and the fourth lines would be different in the last copy:

$$\begin{aligned} l_i^1 &= c_{2i}^1 \cdots c_1^1, \\ l_o^1 &= c_1^1 \cdots c_{2i}^1, \\ r_i^n &= c_{2i+2}^n \cdots c_{2h+1}^n, \\ r_o^n &= c_{2h+1}^n \cdots c_{2i+2}^n. \end{aligned}$$

Basically l_i^j is the *inward* product of the horizontal twists on the *left* hand side of the j^{th} copy, namely their product taken towards the center of the j^{th} copy. Similarly l_o^j is the *outward* product of the twists on the *left* hand side of the j^{th} copy, namely their product taken away from the center of the j^{th} copy. The definitions of r_i^j and r_o^j use the same idea. m^j represents the product of the twists in the *middle* section of the j^{th} copy as it appears in Theorem 1.0.1

The twists $c_{2i}^j, c_{2i}^1, c_{2i+2}^j$ and c_{2i+2}^n should actually be written as $c_{2i_j}^j, c_{2i_1}^1, c_{2i_j+2}^j$ and $c_{2i_n+2}^n$ as the subindex i will be different for each copy but we are not showing this dependence on j to keep the notation simple.

Using the notation described above we can write the positive Dehn twist product for the involution shown in Figure 18:

Theorem 2.1.3. *The positive Dehn twist product for the involution θ shown in Figure 18 is*

$$r_i^n l_i^n r_i^{n-1} l_i^{n-1} x_{n-1} t_{n-1} b_0^n r_o^n l_o^n m^n \cdots m^4 r_i^2 l_i^2 x_2 t_2 b_0^3 r_o^3 l_o^3 m^3 r_i^1 l_i^1 x_1 t_1 b_0^2 r_o^2 l_o^2 m^2 b_0^1 r_o^1 l_o^1 m^1.$$

To reduce the notation in the above expression for θ further let's let

$$\begin{aligned} Y_o^j &= r_o^j l_o^j m^j, \\ Y_i^j &= r_i^j l_i^j, \\ X_j &= x_j t_j. \end{aligned}$$

Then θ can be rewritten as

$$Y_i^n Y_i^{n-1} X_{n-1} b_0^n Y_o^n \cdots b_0^4 Y_o^4 Y_i^2 X_2 b_0^3 Y_o^3 Y_i^1 X_1 b_0^2 Y_o^2 b_0^1 Y_o^1.$$

Using the product notation we obtain:

$$\theta = Y_i^n \prod_{j=2}^n \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1$$

The product sign here will mean multiplication from right to left, contrary to its usual meaning, in agreement with the earlier expressions.

Proof: The proof is by induction. To show the effect of θ on the first copy we set $n = 2$ in the product sign and get the expression

$$Y_i^1 X_1 b_0^2 Y_o^2 b_0^1 Y_o^1,$$

which is equal to

$$Y_i^1 X_1 b_0^1 Y_o^1 b_0^2 Y_o^2,$$

because none of the cycles in $b_0^1 Y_o^1$ intersects any cycle in $b_0^2 Y_o^2$. Therefore what we have is

$$r_i^1 l_i^1 x_1 t_1 b_0^1 r_o^1 l_o^1 m^1 b_0^2 r_o^2 l_o^2 m^2,$$

which can be reduced to

$$r_i^1 l_i^1 x_1 t_1 b_0^1 r_o^1 l_o^1 m^1$$

because the expression $b_0^2 r_o^2 l_o^2 m^2$ has no effect on the bounded surface in Figure 19.

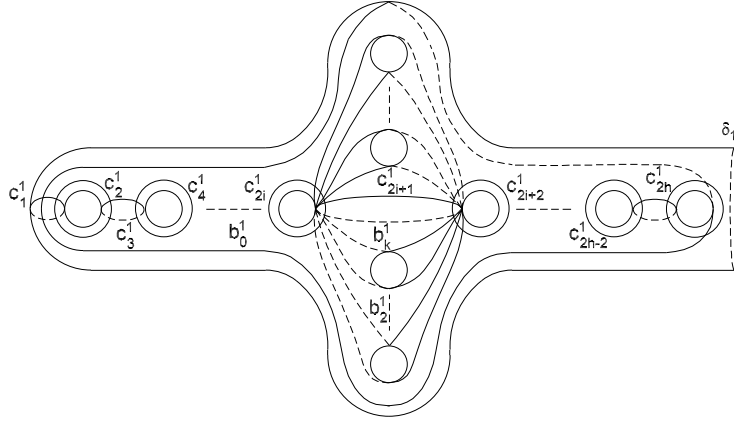


FIGURE 19. Initial step

The explicit version of $r_i^1 l_i^1 x_1 t_1 b_0^1 r_o^1 l_o^1 m^1$ is

$$c_{2i+2}^1 \cdots c_{2h}^1 c_{2i}^1 \cdots c_1^1 x_1 t_1 b_0^1 c_{2h}^1 \cdots c_{2i+2}^1 c_1^1 c_2^1 \cdots c_{2i}^1 b_1^1 b_2^1 \cdots b_{k-1}^1 b_k^1 c_{2i+1}^1,$$

This is a special case of the expression in Proposition 2.1.2 for the surface with one boundary component. Therefore the effect of the above expression on the bounded surface in Figure 19 is that of θ in Proposition 2.1.2.

Now suppose that

$$\prod_{u=2}^j (Y_i^{u-1} X_{u-1} b_0^u Y_o^u) b_0^1 Y_o^1$$

realizes the involution θ on the first $j-1$ copies of the surface in Figure 18. Consider now

$$\prod_{u=2}^{j+1} (Y_i^{u-1} X_{u-1} b_0^u Y_o^u) b_0^1 Y_o^1,$$

which is equal to

$$Y_i^j X_j b_0^{j+1} Y_o^{j+1} \prod_{u=2}^j (Y_i^{u-1} X_{u-1} b_0^u Y_o^u) b_0^1 Y_o^1.$$

The first observation we have to make is, the expression $Y_i^j X_j b_0^{j+1} Y_o^{j+1}$ leaves the first $j-1$ copies with boundary δ_{j-1} unaltered, because all of the twists it contains are about cycles that lie completely to the right of δ_{j-1} . Now, in order to see the effect of the inductive step on the surface in Figure 20 let's release the last term in the product to get

$$Y_i^j X_j b_0^{j+1} Y_o^{j+1} Y_i^{j-1} X_{j-1} b_0^j Y_o^j \prod_{u=2}^{j-1} (Y_i^{u-1} X_{u-1} b_0^u Y_o^u) b_0^1 Y_o^1.$$

The part of this expression that will be effective in the mapping of the j^{th} copy is contained in $Y_i^j X_j b_0^{j+1} Y_o^{j+1} Y_i^{j-1} X_{j-1} b_0^j Y_o^j$.

Using the commutativity relation between the terms that do not intersect we can rewrite this as $Y_i^{j-1} Y_i^j X_j X_{j-1} b_0^j Y_o^j b_0^{j+1} Y_o^{j+1}$, just to bring the terms that we need together. To be precise, the twists contained in $Y_i^j X_j X_{j-1} b_0^j Y_o^j$ are the ones that will realize the effect of θ on the j^{th} copy. Writing them explicitly, we get

$$c_{2i+2}^j \cdots c_{2h}^j c_{2i}^j \cdots c_2^j x_j t_j x_{j-1} t_{j-1} b_0^j c_{2h}^j \cdots c_{2i+2}^j c_2^j \cdots c_{2i}^j b_1^j b_2^j \cdots b_{k-1}^j b_k^j c_{2i+1}^j,$$

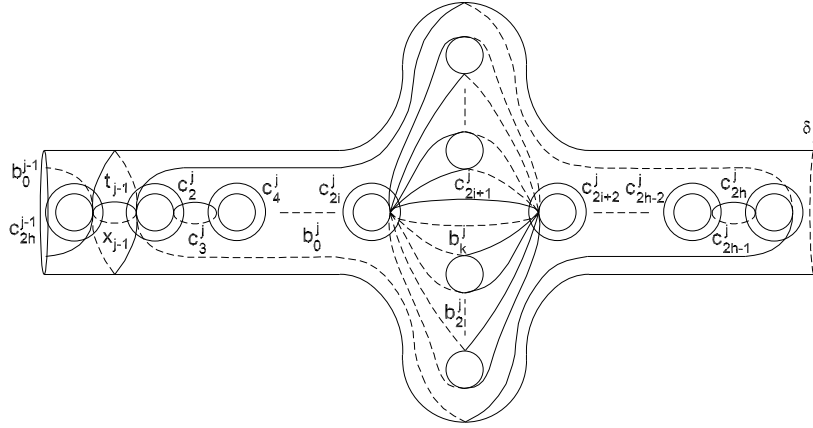


FIGURE 20. Inductive step

which is exactly the expression in Proposition 2.1.2 adapted for the j^{th} copy, with the identifications $c_1 = t_{j-1}, c_{2h+1} = t_j, x_1 = x_{j-1}, x_2 = x_j$. This proves the inductive step.

To complete the proof we need to point out to the mapping of the last copy. Recall the expression for θ

$$\theta = Y_i^n \prod_{j=2}^n \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1.$$

Releasing the last term in the product sign we get

$$Y_i^n Y_i^{n-1} X_{n-1} b_0^n Y_o^n \prod_{j=2}^{n-1} \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1$$

Since Y_i^{n-1} has no effect on the n^{th} copy we have only $Y_i^n X_{n-1} b_0^n Y_o^n$ realizing θ on the last copy. Writing them explicitly we get

$$c_{2i+2}^n \cdots c_{2h}^n c_{2h+1}^n c_{2i}^n \cdots c_2^n x_{n-1} t_{n-1} b_0^n c_{2h+1}^n c_{2h}^n \cdots c_{2i+2}^n c_2^n \cdots c_{2i}^n b_1^n b_2^n \cdots b_{k-1}^n b_k^n c_{2i+1}^n,$$

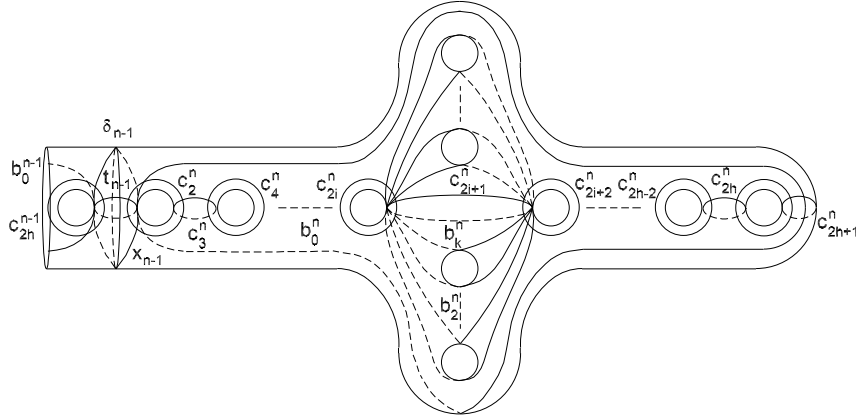


FIGURE 21. Last copy

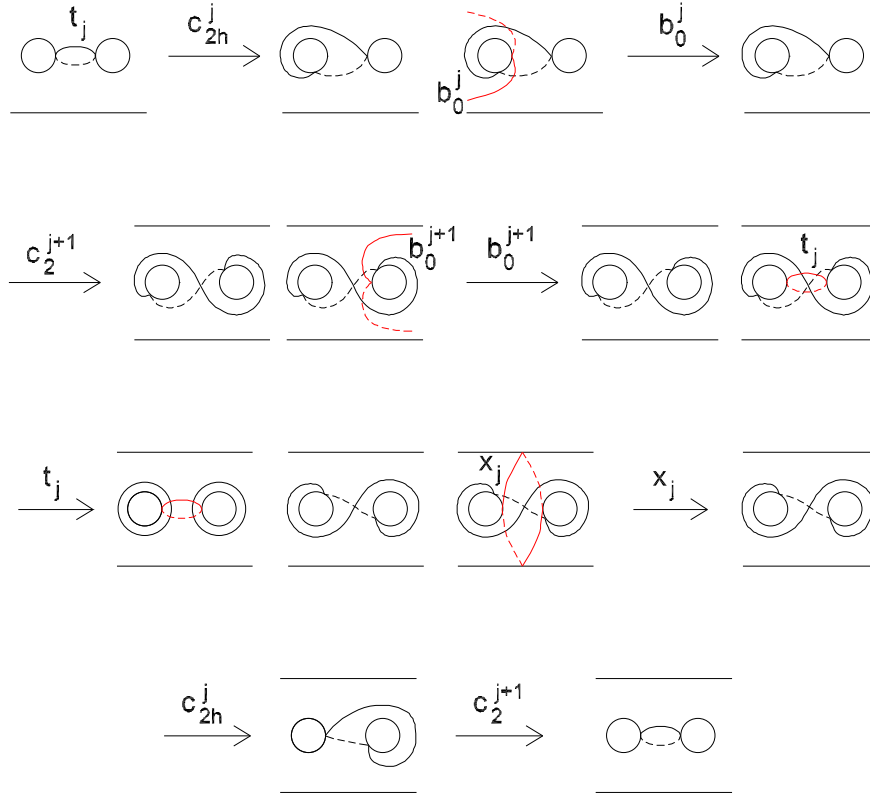
This is, again, a special case of the formula in Proposition 2.1.2 adapted for the surface with one boundary component seen in Figure 21.

Although it is not needed, we will also include the mapping of the cycle t_j in the proof.

Figure 22 shows the mapping of the curve $t_j, j = 1 \dots, n-1$. To understand the steps in that figure let's write

$$\theta = \cdots Y_i^{j+1} X_{j+1} b_0^{j+2} Y_o^{j+2} Y_i^j X_j b_0^{j+1} Y_o^{j+1} Y_i^{j-1} X_{j-1} b_0^j Y_o^j \cdots$$

The first term in the expression for θ that will not miss t_j is $Y_o^j = r_o^j l_o^j m^j$. In fact all the twists in Y_o^j will miss t_j except for the last twist in r_o^j , which is c_{2h}^j . The next twist is b_0^j and it will leave the result of the previous twist unaltered as shown in the first line of Figure 22. So does the expression $Y_i^{j-1} X_{j-1}$ because all the twists they contain miss the same result. The effect of the next term $Y_o^{j+1} = r_o^{j+1} l_o^{j+1} m^{j+1}$ on

FIGURE 22. The mapping of t_j

the current cycle is performed by the twist c_2^{j+1} which is contained in l_o^{j+1} . The result from this twist is seen in the second line of the figure. This movement causes the next twist to miss the current result, namely b_0^{j+1} leaves it unaltered as shown in the second line. The first twist t_j in $X_j = x_j t_j$ has two intersection points with the current cycle and the result from its application is seen in the first half of the third line. The cycle x_j doesn't intersect the result from twisting about t_j , therefore it has no effect on it as indicated in the end of the third line. The following term $Y_i^j = r_i^j l_i^j$ has only one cycle that will intersect the cycle that is missed by x_j in the third line, i.e., c_{2h}^j that lies in r_i^j . Its effect on the current cycle is seen in the beginning of the last line. All the twists contained in the sequence of terms $X_{j+1} b_0^{j+2} Y_o^{j+2}$ following Y_i^j miss the first cycle in the last line. The next cycle that will not miss it is c_2^{j+1} , which is contained in l_i^{j+1} of $Y_i^{j+1} = r_i^{j+1} l_i^{j+1}$. The rest of the twists miss the last cycle in Figure 22, therefore t_j is fixed point-wise under the action of the expression for θ , as expected.

Corollary 2.1.4. *Let θ be expressed as in Theorem 1.0.1. By setting $k = 0$ we obtain the positive Dehn twist expression*

$$i = c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 b_0 c_{2h+1} c_{2h} \cdots c_{2i+2} c_1 c_2 \cdots c_{2i} c_{2i+1}$$

for the hyperelliptic involution.

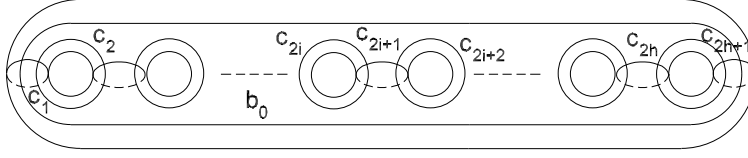


FIGURE 23. A positive Dehn twist expression for the hyperelliptic involution

Proof: We will give an algebraic proof for this fact. First, observe that $b_0 = c_1 \cdots c_{2h}(c_{2h+1})$. Here we abuse the notation and use the same notation for the cycles and the twists. $b_0 = c_1 \cdots c_{2h}(c_{2h+1})$ means that the sequence of twists $c_1 \cdots c_{2h}$ are applied to the cycle c_{2h+1} and the cycle b_0 is obtained as the result of that. Therefore the twist about b_0 is obtained from the twist about c_{2h+1} by conjugation by $c_1 \cdots c_{2h}$ by a well-known fact [1], i.e.,

$$b_0 = c_1 \cdots c_{2h} c_{2h+1} (c_1 \cdots c_{2h})^{-1}.$$

Substituting this in the expression for i stated in Corollary 2.1.4 we obtain

$$c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 c_1 \cdots c_{2h} c_{2h+1} (c_1 \cdots c_{2h})^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_1 c_2 \cdots c_{2i} c_{2i+1},$$

$$c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 c_1 \cdots c_{2h} c_{2h+1} c_{2h}^{-1} \cdots c_1^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_1 c_2 \cdots c_{2i} c_{2i+1}.$$

Recall that c_i and c_j commute if $|i - j| > 1$. Using this we can write

$$c_{2h}^{-1} \cdots c_1^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_1 c_2 \cdots c_{2i} c_{2i+1}$$

as

$$c_{2h}^{-1} \cdots c_{2i+1}^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i}^{-1} \cdots c_1^{-1} c_1 c_2 \cdots c_{2i} c_{2i+1}$$

$$c_{2h}^{-1} \cdots c_{2i+1}^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i+1}$$

Therefore what we had originally is equal to

$$c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 c_1 \cdots c_{2h} c_{2h+1} c_{2h}^{-1} \cdots c_{2i+1}^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i+1}.$$

Another relation we have to remember is the braid relation $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$, from which we can obtain $c_i c_{i+1} c_i^{-1} = c_{i+1}^{-1} c_i c_{i+1}$. Using this multiple times on the expression

$$c_1 \cdots c_{2h} c_{2h+1} c_{2h}^{-1} \cdots c_{2i+1}^{-1}$$

along with the commutativity relation mentioned above, we obtain

$$c_{2h+1}^{-1} \cdots c_{2i+2}^{-1} c_1 \cdots c_{2h} c_{2h+1}.$$

Substituting this in what we have for the original expression now we get

$$c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 c_1 \cdots c_{2h} c_{2h+1} c_{2h}^{-1} \cdots c_{2i+1}^{-1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i+1}$$

$$c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 c_{2h+1}^{-1} \cdots c_{2i+2}^{-1} c_1 \cdots c_{2h} c_{2h+1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i+1}.$$

Using the commutativity relation between the terms $c_{2i} \cdots c_2 c_1$ and $c_{2h+1}^{-1} \cdots c_{2i+2}^{-1}$ we can write the above expression as

$$c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2h+1}^{-1} \cdots c_{2i+2}^{-1} c_{2i} \cdots c_2 c_1 c_1 \cdots c_{2h} c_{2h+1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i+1},$$

which simplifies to

$$c_{2i} \cdots c_2 c_1 c_1 \cdots c_{2h} c_{2h+1} c_{2h+1} c_{2h} \cdots c_{2i+2} c_{2i+1}.$$

If we square this we get

$$c_{2i} \cdots c_1 \underline{c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1} c_{2i}} \cdots c_1 c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1}.$$

The underlined portion is the well-known expression for i . Also using the fact that i commutes with c_i , the above expression becomes

$$i c_{2i} \cdots c_1 c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1}.$$

Now the question reduces to showing

$$i c_{2i} \cdots c_1 c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1} = 1.$$

We will obtain that result by going backwards from the relation $i^2 = 1$, by first writing it as

$$i c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1} c_{2i} \cdots c_1 = 1,$$

then multiplying by c_1^{-1} on the right,

$$i c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1} c_{2i} \cdots c_2 = c_1^{-1},$$

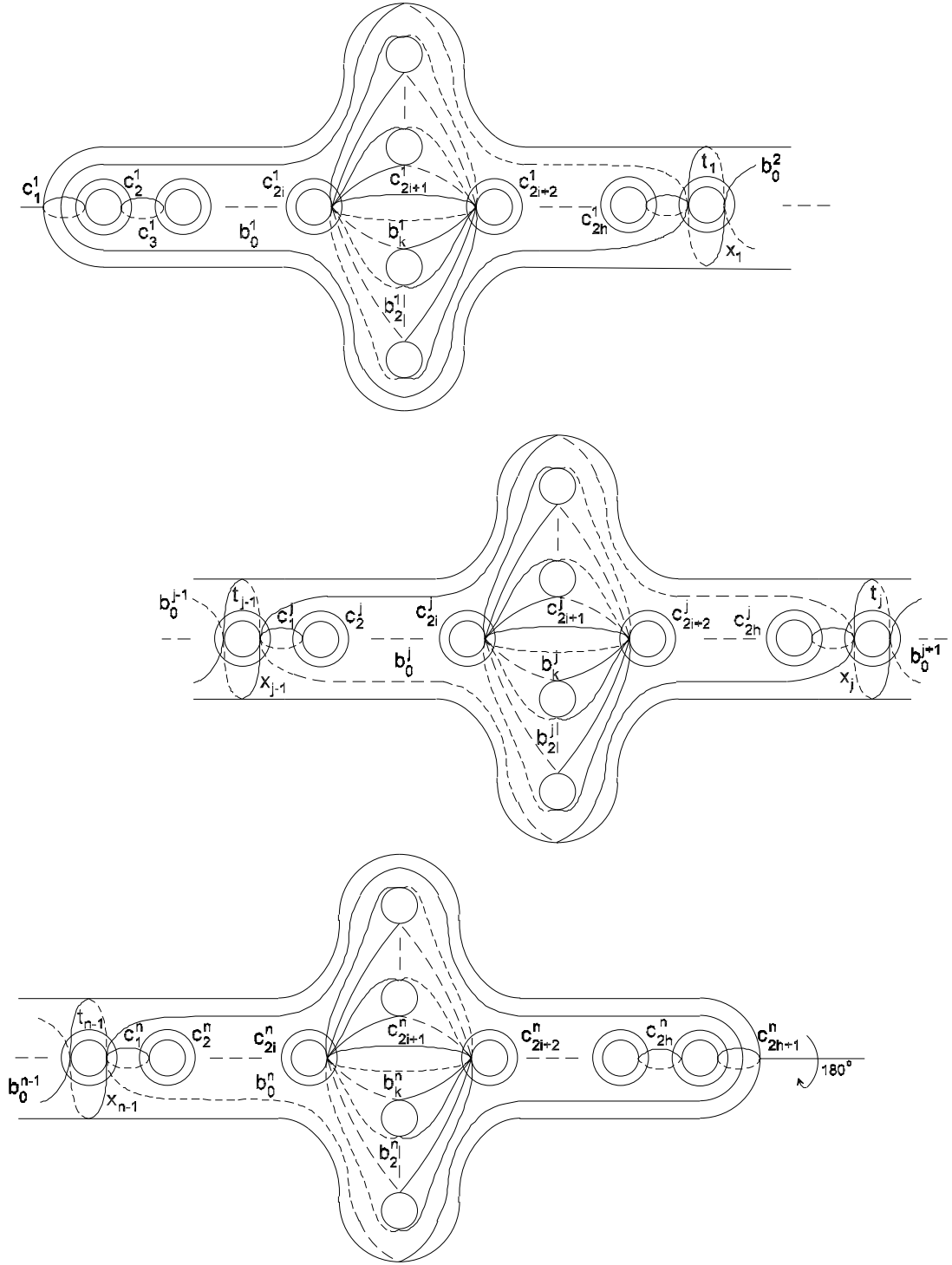
and then multiplying by c_1 on the left

$$i c_1 c_1 \cdots c_{2h+1} c_{2h+1} \cdots c_{2i+1} c_{2i} \cdots c_2 = 1$$

and repeating the same procedure $2i$ times.

An alternate expression for θ using a slightly different set of cycles is obtained by gluing n copies of bounded surfaces in Figure 4 together along tori with two boundary components. Figure 24 demonstrates the set of cycles that are used in that expression. The need for this expression emerges from the fact that it is necessary to have at least two holes between two copies when they are glued along four-holed spheres, as seen in Figures 27 and 28. The alternate expression allows us to have only one hole between two adjacent copies and it is very similar to the one given in Theorem 2.1.3:

$$\theta = Y_i^n \prod_{j=2}^n \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1,$$


 FIGURE 24. An alternate expression for θ

where

$$\begin{aligned} Y_o^j &= r_o^j l_o^j m^j, \\ Y_i^j &= r_i^j l_i^j, \\ X_j &= x_j t_j, \end{aligned}$$

and

$$\begin{aligned} l_i^j &= c_{2i}^j \cdots c_1^j, \\ l_o^j &= c_1^j \cdots c_{2i}^j, \\ r_i^j &= c_{2i+2}^j \cdots c_{2h+1}^j, \\ r_o^j &= c_{2h+1}^j \cdots c_{2i+2}^j, \\ m^j &= b_1^j \cdots b_k^j c_{2i+1}^j. \end{aligned}$$

The proof of this fact also uses induction and it is essentially based on the simple bounded case that is similar to the one in Proposition 2.1.2. After modifying Figure 4 slightly, we obtain Figure 25

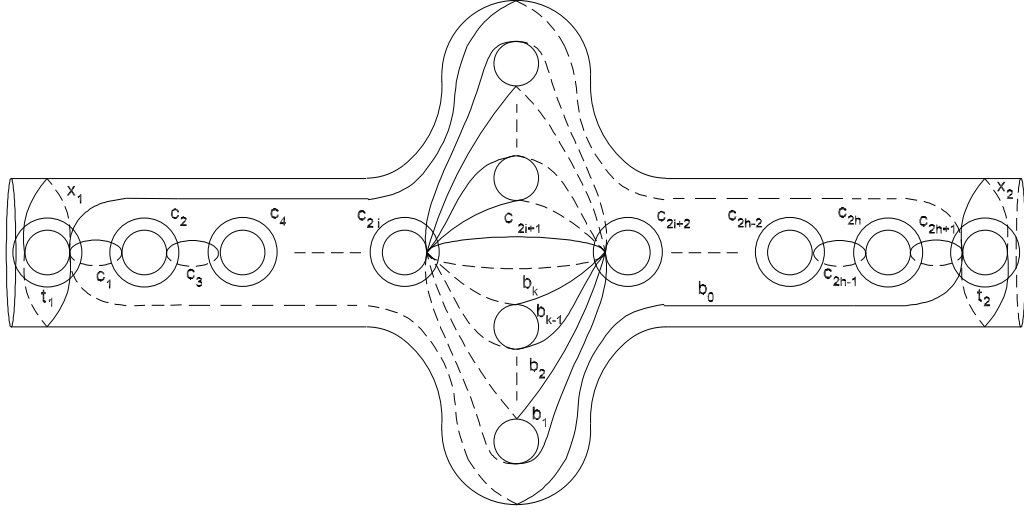
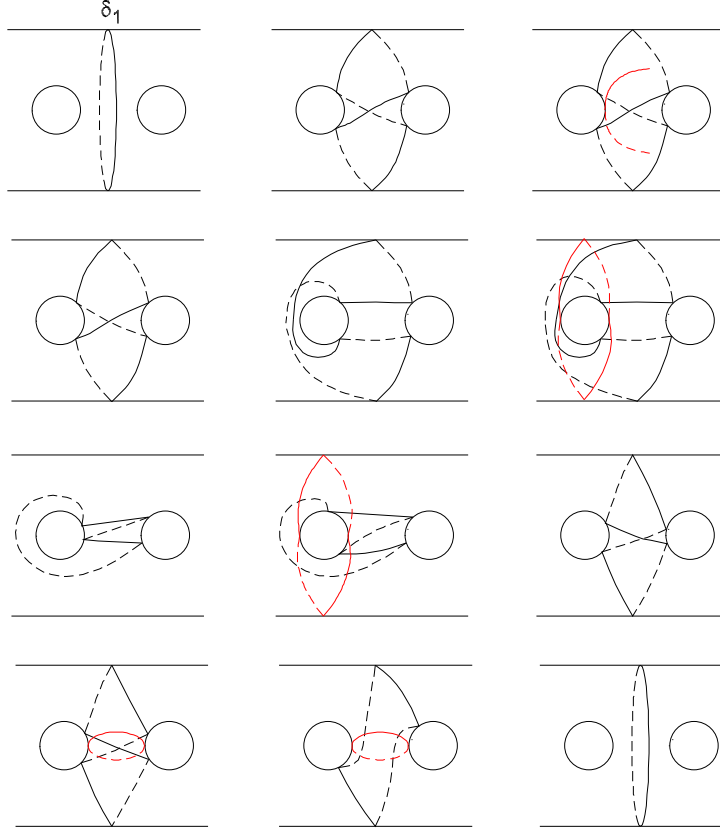


FIGURE 25. Alternate cycles for θ defined on the bounded surface in Figure 4

and hence the expression

$$\theta = c_{2i+2} \cdots c_{2h+1} c_{2i} \cdots c_1 x_2 t_2 x_1 t_1 b_0 c_{2h+1} \cdots c_{2i+2} c_1 \cdots c_{2i} b_1 b_2 \cdots b_{k-1} b_k c_{2i+1}$$

that replaces the one in Proposition 2.1.2. We will not give a detailed proof for this last expression, instead just provide the mapping of the boundary component δ_1 in Figure 4. One has to mimic the steps that is involved in the mapping of the other cycles in the proof of Proposition 2.1.2 by accommodating the slight modifications as needed.


 FIGURE 26. The mapping of δ_1

In the first line of Figure 26 we see the effect of c_1 on δ_1 first because all the cycles that come before c_1 miss δ_1 . The next cycle, b_0 , misses the result from that as seen in the end of the first and the beginning of the second line. After that, the twist about t_1 takes place, which is not demonstrated in two steps, even though it intersects the cycle it twists twice. The result from that has intersection number 2 with x_1 and the twist about x_1 is shown in two steps in the end of the second line and all of the third line. Following twists miss completely the cycle that is in the end of the third line and the twist about c_1 brings that cycle back to δ_1 .

A corollary to the expression for θ in Theorem 2.1.3 and its alternate form would be setting $k = 0$ to obtain some new expressions for the hyperelliptic involution. All we have to do is redefine Y_o^j as $Y_o^j = r_o^j l_o^j c_{2i+1}^j$ without changing the expression

$$Y_i^n \prod_{j=2}^n \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1,$$

for θ .

3. APPLICATIONS

In this section we will determine the homeomorphism type of the genus g Lefschetz fibration

$$X \longrightarrow S^2$$

described by the word $\theta^2 = 1$ in the mapping class group M_g , where θ is as defined in Theorem 2.1.3.

Consider the surface in Figure 18. Let k_j be the j^{th} vertical genus, the total genus of the central part of the j^{th} copy, and let $h_j = l_j + r_j$ be the j^{th} horizontal genus, namely the sum of the j^{th} left genus l_j and the j^{th} right genus r_j . Let $k = \sum k_j$ be the vertical genus and $h = \sum h_j$ be the horizontal genus. If we denote the total genus by g then $g = h + k$.

To find the total number of cycles contained in θ let's recall that

$$\theta = Y_i^n \prod_{j=2}^n \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1,$$

where

$$\begin{aligned} Y_o^j &= r_o^j l_o^j m^j, \\ Y_i^j &= r_i^j l_i^j, \\ X_j &= x_j t_j, \end{aligned}$$

and

$$\begin{aligned} l_i^j &= c_{2i}^j \cdots c_2^j, \\ l_o^j &= c_2^j \cdots c_{2i}^j, \\ r_i^j &= c_{2i+2}^j \cdots c_{2h}^j, \\ r_o^j &= c_{2h}^j \cdots c_{2i+2}^j, \\ m^j &= b_1^j \cdots b_k^j c_{2i+1}^j. \end{aligned}$$

The first and the last copies would differ in the first two and the following two lines of definitions above, respectively:

$$\begin{aligned} l_i^1 &= c_{2i}^1 \cdots c_1^1, \\ l_o^1 &= c_1^1 \cdots c_{2i}^1, \\ r_i^n &= c_{2i+2}^n \cdots c_{2h+1}^n, \\ r_o^n &= c_{2h+1}^n \cdots c_{2i+2}^n. \end{aligned}$$

m^j consists of $k_j + 1$ cycles. Both l_i^j and l_o^j consist of $2i - 2 + 1 = 2i - 1$ cycles for $j \neq 1$ and l_i^1 and l_o^1 consist of $2i - 1 + 1 = 2i$ cycles. Likewise, both r_i^j and r_o^j consist of $2h_j - (2i + 2) + 1 = 2h_j - 2i - 1$ cycles for $j \neq n$ and r_i^n and r_o^n consist of $2h_n + 1 - (2i + 2) + 1 = 2h_n - 2i$ cycles. Therefore Y_o^j consists of $y_o^j = 2h_j - 2i - 1 + 2i - 1 + k_j + 1 = k_j + 2h_j - 1$ cycles for $j \neq 1, n$, Y_o^1 consists of $y_o^1 = 2h_1 - 2i - 1 + 2i + k_1 + 1 = k_1 + 2h_1$ cycles, and Y_o^n consists of $y_o^n = 2h_n - 2i + 2i - 1 + k_n + 1 = k_n + 2h_n$ cycles. Y_i^j has $y_i^j = 2h_j - 2i - 1 + 2i - 1 = 2h_j - 2$ cycles for $j \neq 1, n$, Y_i^1 has $y_i^1 = 2h_1 - 2i - 1 + 2i = 2h_1 - 1$ cycles, and Y_i^n has $y_i^n = 2h_n - 2i + 2i - 1 = 2h_n - 1$ cycles. Clearly X_j consists of 2 cycles.

In the above computations, too, we ignored the dependence of i on j and did not write $2i_j$ instead of $2i$ in order not to make the computations more complicated because they cancel out anyway.

Now, using the lengths of each group of twists computed above we determine that

$$Y_i^n \prod_{j=2}^n \left(Y_i^{j-1} X_{j-1} b_0^j Y_o^j \right) b_0^1 Y_o^1,$$

consists of

$$\begin{aligned} & y_i^n + \sum_{j=2}^n \left(y_i^{j-1} + 2 + 1 + y_o^j \right) + 1 + y_o^1 \\ &= y_i^n + \sum_{j=2}^n y_i^{j-1} + 3(n-2+1) + \sum_{j=2}^n y_o^j + 1 + y_o^1 \end{aligned}$$

cycles. Rearranging the indices and simplifying we get

$$y_i^n + \sum_{j=1}^{n-1} y_i^j + 3(n-1) + \sum_{j=2}^n y_o^j + 1 + y_o^1$$

Releasing the first term of the first sum and the last term of the second sum gives

$$y_i^n + y_i^1 + \sum_{j=2}^{n-1} y_i^j + 3(n-1) + y_o^n + \sum_{j=2}^{n-1} y_o^j + 1 + y_o^1,$$

which is equal to

$$y_i^1 + y_o^1 + y_i^n + y_o^n + \sum_{j=2}^{n-1} (y_i^j + y_o^j) + 3(n-1) + 1.$$

Now substituting the value of each term and simplifying we obtain

$$\begin{aligned} & 2h_1 - 1 + k_1 + 2h_1 + 2h_n - 1 + k_n + 2h_n + \sum_{j=2}^{n-1} (2h_j - 2 + k_j + 2h_j - 1) + 3(n-1) + 1 \\ &= 4h_1 + k_1 + 4h_n + k_n + \sum_{j=2}^{n-1} (4h_j + k_j) - 3(n-1-2+1) + 3(n-1) - 1 \\ &= \sum_{j=1}^n (4h_j + k_j) - 3(n-2) + 3(n-1) - 1 \\ &= 4 \sum_{j=1}^n h_j + \sum_{j=1}^n k_j + 6 - 3n + 3n - 3 - 1 \\ &= 4h + k + 2 \end{aligned}$$

Therefore the word $\theta^2 = 1$ consists of $2(4h + k + 2) = 8h + 2k + 4$ twists.

Since all the twists are about non-separating cycles, the Lefschetz fibration defined by the word $\theta^2 = 1$ has $8h + 2k + 4$ irreducible fibers. This allows us to compute the Euler characteristic of the 4– manifold X using the formula

$$\chi(X) = 2(2 - 2g) + \text{number of singular fibers}$$

for Lefschetz fibrations, which is

$$\begin{aligned} 2(2 - 2g) + 8h + 2k + 4 &= 4 - 4g + 8h + 2k + 4 = 4 - 4(h + k) + 8h + 2k + 4 \\ &= 8 + 4h - 2k \end{aligned}$$

in our case.

The other homeomorphism invariant that we will compute is the signature $\sigma(X)$ of the 4– manifold X .

Using the algorithm described in [2] we wrote a Matlab program that computes the signature of the Lefschetz fibration described by the word $\theta^2 = 1$.

The input for the program is the left, right, and the vertical genus of each copy that is glued together to form the surface Σ on which θ is defined. The following are two examples that demonstrate how the shape of the surface is coded into a sequence of numbers, which are used as the inputs for the program.

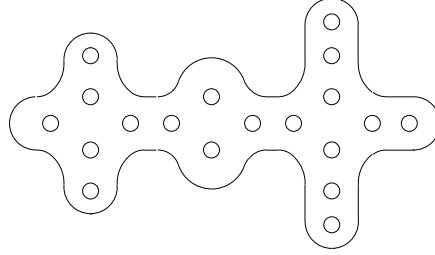


FIGURE 27. A surface with input (1 4 1,1 2 1,1 6 2)

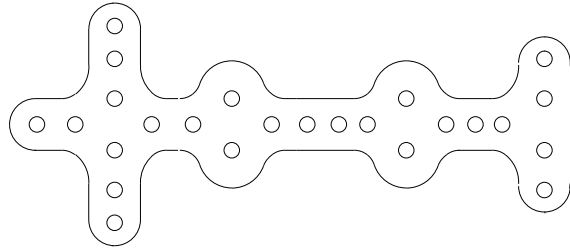


FIGURE 28. A surface with one possible input (2 6 1,1 2 2,2 2 1,2 4 0)

(1 4 1,1 2 1,1 6 2): -32

The above computations, along with many others that we do not include here, point out to the fact that the signature depends only on h , i.e., it is independent of k . A quick check suggests that $\sigma(X) = -4(h+1)$ for the above computations. We conjecture that this is true in general, namely the signature of the Lefschetz fibration given by the word $\theta^2 = 1$, where θ is as defined in Theorem 2.1.3, is $-4(h+1)$.

For $\chi(X) = 8 + 4h - 2k$ and $\sigma(X) = -4(h+1)$, we obtain

$$\begin{aligned} c_1^2(X) &= 3\sigma(X) + 2\chi(X) \\ &= 3(-4h - 4) + 2(8 + 4h - 2k) \\ &= -4h - 4k + 4 \\ &= -4(g - 1) \end{aligned}$$

and

$$\begin{aligned} \chi_h(X) &= \frac{1}{4}(\sigma(X) + \chi(X)) \\ &= \frac{1}{4}(8 + 4h - 2k - 4h - 4) \\ &= \frac{1}{4}(4 - 2k) = 1 - k/2 \end{aligned}$$

Recall that k is even. $\chi_h(X)$ makes sense here because X has almost complex structure.

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